

# Multiple solutions for Kirchhoff equations under the partially sublinear case

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## Abstract

In this paper, we prove the infinitely many solutions to a class of sublinear Kirchhoff type equations by using an extension of Clark's theorem established by Zhaoli Liu and Zhi-Qiang Wang.

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*Keywords:* Kirchhoff type equations; Clark's theorem; Infinitely many solutions

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## 1. Introduction and main results

In this paper we study the existence and multiplicity of solutions for the following Kirchhoff type equations:

$$\left(a + \int_{\mathbb{R}^N} |\nabla u|^2 + b \int_{\mathbb{R}^N} u^2\right) [-\Delta u + bu] = K(x)f(x, u), \text{ in } \mathbb{R}^3, \quad (1.1)$$

where  $a, b$  are positive constants.

When  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^3$ , the problem

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

has been many papers concerned. Perera and Zhang [1] considered the case where  $f(x, \cdot)$  is asymptotically linear at 0 and asymptotically 4-linear at infinity. They obtained a nontrivial solution of the problems by using the Yang index and critical group. Then, in [1] they considered the cases where  $f(x, \cdot)$  is 4-sublinear, 4-superlinear and asymptotically 4-linear at infinity. By various assumption on  $f(x, \cdot)$  near 0, they obtained multiple and sign changing solutions. Cheng and Wu [3], Ma and Rivera [4] studied the existence of positive solutions of (1.2) and He and Zou [5] obtained the existence of infinitely many positive solutions of (1.2), respectively; Mao and Luan [6] obtained the existence of signed and sign-changing solutions for the problem (1.2) with asymptotically

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4-linear bounded nonlinearity via variational methods and invariant sets of descent flow; Sun and Tang [7] studied the existence and multiplicity results of nontrivial solutions for the problem (1.2) with the weaker monotony and 4-superlinear nonlinearity. For (1.2), Sun and Liu [8] considered the cases where the nonlinearity is superlinear near zero but asymptotically 4-linear at infinity, and the nonlinearity is asymptotically linear near zero but 4-superlinear at infinity. By computing the relevant critical groups, they obtained nontrivial solutions via Morse theory.

Comparing with (1.1) and (1.2),  $\mathbb{R}^3$  in place of the bounded domain  $\Omega \subset \mathbb{R}^3$ . This make that the study of the problem (1.1) is more difficult and interesting. Wu [11] considered a class of Schrödinger Kirchhoff-type problem in  $\mathbb{R}^N$  and a sequence of high energy solutions are obtained by using a symmetric Mountain Pass Theorem. In [12], Alves and Figueiredo study a periodic Kirchhoff equation in  $\mathbb{R}^N$ , they get the nontrivial solution when the nonlinearity is in subcritical case and critical case. Liu and He [13] get multiplicity of high energy solutions for superlinear Kirchhoff equations in  $\mathbb{R}^3$ . Li, Li and Shi in [15] proved the existence of a positive solution to a Kirchhoff type problem on  $\mathbb{R}^N$  by using variational methods and cut-off functional technique.

In [9], Jin and Wu in consider the following problem:

$$\begin{cases} -(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u + u = f(x, u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.3)$$

where constants  $a > 0, b > 0$ ,  $N = 2$  or  $3$  and  $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ .

By using the Fountain Theorem, they obtained the following theorem.

**Theorem A** [9] *Assume that the following conditions hold:*

*If the following assumptions are satisfied,*

*(H<sub>1</sub>)  $f(x, u) = o(|u|)$  as  $|u| \rightarrow 0$  uniformly for any  $x \in \mathbb{R}^N$ .*

*(H<sub>2</sub>) There are constants  $1 < p < 2^* - 1$  and  $c > 0$  such that*

$$|f(x, u)| \leq c(1 + |u|^p), \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R},$$

*where*

$$2^* - 1 = \begin{cases} \frac{N+2}{N-2}, & N \geq 3; \\ +\infty, & N = 1, 2. \end{cases}$$

*(H<sub>3</sub>) There exists  $\mu > 4$  such that*

$$\mu F(x, u) = \mu \int_0^u f(x, s) ds \leq u f(x, u), \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

*(H<sub>4</sub>)*

$$\inf_{x \in \mathbb{R}^N, |u|=1} F(x, u) > 0$$

*(H<sub>5</sub>)  $f(gx, u) = f(x, u)$  for each  $g \in O(N)$  and for each  $(x, u) \in \mathbb{R}^N \times \mathbb{R}$ , where  $O(N)$  is the group of orthogonal transformations on  $\mathbb{R}^N$ .*

(H<sub>6</sub>)  $f(x, -u) = -f(x, u)$  for any  $(x, u) \in \mathbb{R}^N \times \mathbb{R}$ .

Then problem (1.3) has a sequence  $\{u_k\}$  of radial solutions.

Recently, the authors obtained an extension of Clark's theorem as follows.

**Theorem B** [10] *Let  $X$  be a Banach space,  $\Phi \in C^1(X, \mathbb{R})$ . Assume  $\Phi$  is even and satisfies the (PS) condition, bounded from below, and  $\Phi(0) = 0$ . If for any  $k \in \mathbb{N}$ , there exists a  $k$ -dimensional subspace  $X^k$  of  $X$  and  $\rho_k > 0$  such that  $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$ , where  $S_\rho = \{u \in X \mid \|u\| = \rho\}$ , then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points  $\{u_k\}$  satisfying  $\Phi(u_k) < 0$  for all  $k$  and  $\|u_k\| \rightarrow 0$  as  $k \rightarrow \infty$ .*

(ii) *There exists  $r > 0$  such that for any  $0 < a < r$  there exists a critical point  $u$  such that  $\|u\| = a$  and  $\Phi(u) = 0$ .*

In this paper, we consider the multiple solutions for Kirchhoff equations under the partially sublinear case by using the Theorem C. Our main result is as follows.

**Theorem 1.1** *Assume that  $f$  satisfies (B<sub>3</sub>) and the following conditions:*

(f<sub>1</sub>) *There exist  $\delta > 0$ ,  $1 \leq \gamma < 2$ ,  $C > 0$  such that  $f \in C(\mathbb{R}^3 \times [-\delta, \delta], \mathbb{R})$  and  $|f(x, z)| \leq C|z|^{\gamma-1}$ ;*

(f<sub>2</sub>)  *$\lim_{z \rightarrow 0} F(x, z)/|z|^2 = +\infty$  uniformly in some ball  $B_r(x_0) \subset \mathbb{R}^3$ , where  $F(x, z) = \int_0^z f(x, s)ds$ .*

(f<sub>3</sub>)  *$K : \mathbb{R}^3 \rightarrow \mathbb{R}^+$  is a positive continuous function such that  $K \in L^{2/(1-\gamma)}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ .*

*Then (1.1) possesses infinitely many solutions  $\{u_k\}$  such that  $\|u_k\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow \infty$ .*

**Remark 1.1.** Throughout the paper we denote by  $C > 0$  various positive constants which may vary from line to line and are not essential to the problem.

The paper is organized as follows: in Section 2, some preliminary results are presented. Section 3 is devoted to the proof of Theorem 1.1.

## 2. Preliminary

In this Section, we will give some notations and Lemma that will be used throughout this paper.

Let  $H^1 = H^1(\mathbb{R}^3)$  be the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + buvdx], \quad \|u\| = (u, u)^{1/2}.$$

Moreover, we denote the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{D^1} = \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}$$

by  $D^1 = D^1(\mathbb{R}^3)$ . To avoid lack of compactness, we need consider the set of radial functions as follows:

$$H = H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) | u(x) = u(|x|)\}.$$

Here we note that the continuous embedding  $H \hookrightarrow L^q(\mathbb{R}^3)$  is compact for any  $q \in (2, 6)$ .

Define a functional

$$J_1(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 - \int_{\mathbb{R}^3} K(x)F(x, u), \quad u \in H.$$

Then we have from  $(f_1)$  that  $J_1$  is well defined on  $H$  and is of  $C^1$ , and

$$(J_1(u), v) = a(u, v) + \|u\|^2(u, v) - \int_{\mathbb{R}^3} K(x)f(x, u)v, \quad u, v \in H.$$

It is standard to verify that the weak solutions of (1.1) correspond to the critical points of the functional  $J_1$ .

### 3. Proofs of the main result

**Proof of Theorem 1.1.** Choose  $\hat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$  such that  $\hat{f}$  is odd in  $u \in \mathbb{R}$ ,  $\hat{f}(x, u) = f(x, u)$  for  $x \in \mathbb{R}^N$  and  $|u| < \delta/2$ , and  $\hat{f}(x, u) = 0$  for  $x \in \mathbb{R}^N$  and  $|u| > \delta$ . In order to obtain solutions of (1.1) we consider

$$\left(a + \int_{\mathbb{R}^N} |\nabla u|^2 + b \int_{\mathbb{R}^N} u^2\right) [-\Delta u + bu] = K(x)\hat{f}(x, u), \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

Moreover, (3.1) is variational and its solutions are the critical points of the functional defined in  $H$  by

$$J(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\|u\|^4 - \int_{\mathbb{R}^3} K(x)\hat{F}(x, u)dx.$$

From  $(f_1)$ , it is easy to check that  $J$  is well defined on  $H$  and  $J \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$  (see [?] for more detail), and

$$J'(u)v = a(u, v) + \|u\|^2(u, v) - \int_{\mathbb{R}^3} K(x)\hat{f}(x, u)v dx, \quad v \in H.$$

Note that  $J$  is even, and  $J(0) = 0$ . For  $u \in H^1(\mathbb{R}^3)$ ,

$$\int_{\mathbb{R}^3} K(x)|\hat{F}(x, u)|dx \leq C \int_{\mathbb{R}^3} K(x)|u|^\gamma dx \leq C\|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^3)}^\gamma \leq C\|u\|^\gamma.$$

Hence, it follows from Lemma 2.1 that

$$J(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^\gamma, \quad u \in H. \quad (3.2)$$

We now use the same ideas to prove the (PS) condition. Let  $\{u_n\}$  be a sequence in  $H$  so that  $J(u_n)$  is bounded and  $J'(u_n) \rightarrow 0$ . We shall prove that  $\{u_n\}$  converges. By (3.2),

we claim that  $\{u_n\}$  is bounded. Assume without loss of generality that  $\{u_n\}$  converges to  $u$  weakly in  $H$ . Observe that

$$\begin{aligned}\langle J'(u_n) - J'(u), u_n - u \rangle &= a\|u_n - u\|^2 + \|u_n\|^2\|u_n - u\|^2 \\ &\quad + (\|u_n\|^2 - \|u\|^2)(u, u_n - u) \\ &\quad - \int_{\mathbb{R}^3} K(x)(\hat{f}(x, u_n) - \hat{f}(x, u))(u_n - u)dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}a\|u_n - u\|^2 &\leq \langle J'(u_n) - J'(u), u_n - u \rangle - (\|u_n\|^2 - \|u\|^2)(u, u_n - u) \\ &\quad + \int_{\mathbb{R}^3} K(x)(\hat{f}(x, u_n) - \hat{f}(x, u))(u_n - u)dx. \\ &\equiv I_1 + I_2 + I_3,\end{aligned}$$

It is clear that  $I_1 \rightarrow 0$  and  $I_2 \rightarrow 0$  as  $n \rightarrow \infty$ . In the following, we will estimate  $I_3$ , by using (f<sub>3</sub>), for any  $R > 0$ ,

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)|\hat{f}(x, u_n) - \hat{f}(x, u)||u_n - u|dx \\ &\leq C \int_{\mathbb{R}^3 \setminus B_R(0)} K(x)(|u_n|^\gamma + |u|^\gamma)dx + C \int_{B_R(0)} (|u_n|^{\gamma-1} + |u|^{\gamma-1})|u_n - u|dx \\ &\leq C \left( \|u_n\|_{L^2(\mathbb{R}^3 \setminus B_R(0))}^\gamma + \|u\|_{L^2(\mathbb{R}^3 \setminus B_R(0))}^\gamma \right) \|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3 \setminus B_R(0))} \\ &\quad + C \left( \|u_n\|_{L^\gamma(B_R(0))}^{\gamma-1} + \|u\|_{L^\gamma(B_R(0))}^{\gamma-1} \right) \|u_n - u\|_{L^\gamma(B_R(0))} \\ &\leq C \|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3 \setminus B_R(0))} + C \|u_n - u\|_{L^\gamma(B_R(0))},\end{aligned}$$

which implies

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} K(x)|\hat{f}(x, u_n) - \hat{f}(x, u)||u_n - u|dx = 0.$$

Therefore,  $\{u_n\}$  converges strongly in  $H$  and the (PS) condition holds for  $J$ . By (f<sub>2</sub>) and (f<sub>3</sub>), for any  $L > 0$ , there exists  $\delta = \delta(L) > 0$  such that if  $u \in C_0^\infty(B_r(x_0))$  and  $|u|_\infty < \delta$  then  $K(x)\hat{F}(x, u(x)) \geq L|u(x)|^2$ , and it follows from Lemma 2.1 that

$$J(u) \leq \frac{a}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 - L\|u\|_{L^2(\mathbb{R}^3)}^2.$$

This implies, for any  $k \in \mathbb{N}$ , if  $X^k$  is a  $k$ -dimensional subspace of  $C_0^\infty(B_r(x_0))$  and  $\rho_k$  is sufficiently small then  $\sup_{X^k \cap S_{\rho_k}} J(u) < 0$ , where  $S_\rho = \{u \in \mathbb{R}^3 \mid \|u\| = \rho\}$ . Now we apply Theorem C to obtain infinitely many solutions  $\{u_k\}$  for (3.1) such that

$$\|u_k\| \rightarrow 0, \quad k \rightarrow \infty. \quad (3.3)$$

Finally we show that  $\|u_k\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $u$  be a solution of (3.1) and  $\alpha > 0$ . Let  $M > 0$  and set  $u^M(x) = \max\{-M, \min\{u(x), M\}\}$ . Multiplying both sides of (3.1) with  $|u^M|^\alpha u^M$  implies

$$\frac{4a}{(\alpha + 2)^2} \int_{\mathbb{R}^3} |\nabla |u^M|^{\frac{\alpha}{2}+1}|^2 dx \leq C \int_{\mathbb{R}^3} |u^M|^{\alpha+1} dx.$$

By using the iterating method in [10], we can get the following estimate

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C_1 \|u\|_{L^6(\mathbb{R}^3)}^\nu,$$

where  $\nu$  is a number in  $(0, 1)$  and  $C_1 > 0$  is independent of  $u$  and  $\alpha$ . By (3.3) and Sobolev imbedding Theorem[14], we derive that  $\|u_k\|_{L^\infty(\mathbb{R}^3)} \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore,  $u_k$  are the solutions of (1.1) as  $k$  sufficiently large. The proof is completed.  $\square$

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